An Explicit Finite Difference Scheme for 1D Navier- Stokes Eqauation

M.A.K.Azad, L.S. Andallah

Abstract- This paper concerns with the numerical solution of one dimensional Navier-Stokes equation (1D NSE) $u_t + uu_x = -p_x + \frac{1}{\text{Re}} u_{xx}$ for $-p_x = e^{-at-b}$ using Orlowski and Sobczyk transformation (OST). The transformation reduces the NSE into the Burgers equation .We study an explicit finite difference scheme (FDS) for the numerical solution of the reduced 1D NSE as Burgers equation and study stability condition for the scheme. We determine the stability condition as the numerical scheme for the reduced model Burgers equation is the same as that of original Burgers equation. Accuracy and numerical feature of convergence of the explicit scheme is presented by estimating their relative errors. We determine the computational time to implement numerical scheme for reduced NSE and numerical scheme for original NSE for making a comparison.

Key-words- Burgers' equation, Cole-Hopf transformation, Diffusion equation, FDS, NSE, Orlowski and Sobczyk transformation (OST), Reynolds number.

1 INTRODUCTION

The Navier-Stokes equations are one of the most important, beautiful, potentially lucrative governing equations in fluid dynamics which describe the motion of fluid substances. The exact solutions for the NSEs can be obtained are of particular cases. The NSEs are non-linear; there cannot be a general method to solve analytically the full equations. It still remains one of the open problems in the mathematical physics. Exact solutions on the other hand are very important for many reasons. They provide reference solutions to verify the accuracies of many approximate methods. It is thus an important issue to solve 1D NSE analytically as well as numerically.

Numerical simulation of fluid flow has been a major topic of research for the past few decades. Computational Fluid Dynamics (CFD) is one of the prominent physical disciplines that involve the description of the fluid flow in terms of the mathematical models that include convective and diffusive transport of some variables. These mathematical models consist of a set of governing equations in the form of ordinary and partial differential equations. Over the years, the FDS is frequently used in CFD.

Numerical solution of 1D NSE is very important. Analytical solution of 1D NSE can be obtained for particular form of pressure gradient, therefore any numerical technique can be compared with the analytical solution and thus validate the technique.

In order to understand the non-linear phenomenon of NSE, one needs to study 1D NSE as a simplification of full NSE. Nevertheless it incorporates all the main mathematical features of the NSE. Applying OST we have reduced 1D NSE to viscous Burgers equation and we have solved viscous Burgers equation analytically by using CHT. So a number of analytical and numerical studies on 1D NSE and 1D viscous Burgers' equation have been conducted to solve the governing equation analytically and numerically [1],[2],[3],[4],[5],[6],[7],[17].

www.coolissues.com/mathematics/NS [1] studied on analytic solution of 1D Navier-Stokes type equation including and excluding pressure term. Neijib Smaoui [2] studied numerically the long-time dynamics of a system of reaction-diffusion equation that arise from the viscous Burgers equation which is 1D NSE without pressure gradient. Hans J. Wospakrik* and Freddy P. Zen+ [3] presented the solution of the initial value problem of the corresponding linear heat type equation using the Feymann-Kac path integral formulation. A. Orlowski and K. Soczyk [4] presented a transformation to make inhomogeneous Burgers equation to homogeneous form. Ronobir C. Sarker, L.S. Andallah and J. Akhter [5] studied on analytic solution of viscous Burgers equation as an IVP. Due to the complexity of the analytical solution they studies explicit and implicit finite difference schemes for viscous Burgers equation and determined stability condition for both schemes. T. Yang and J.M. McDonough [6] presented exact solution to a 1D Burgers' equation which exhibits erratic turbulent-like behavior. They compared time series of the exact solution with physical experimental data. He also mentioned the governing equation is analogous to 1D Navier-Stokes equation. They also proposed a model to provide a good tool for testing numerical algorithms. J Qian [7] presented numerical experiments on one dimensional model turbulence. M.A.K. Azad and L.S. Andallah [17] studied an analytical solution of 1D Navier-Stokes equation.

In this paper, we present the numerical solution of 1D NSE by using FDS for the reduced 1D NSE. FDS for original 1D NSE is studied. We analyze the pressure gradient and Reynolds number effect in the numerical solution. We discuss the numerical stability of the FDS for the reduced NSE. Then we find relative errors to determine the accuracy of the numerical methods. We determine the computational time to implement numerical scheme for reduced NSE and numerical scheme for original NSE for making a comparison.

2 GOVERNING EQUATION

Using the dimensionless definitions [8], [17]

$$x=\frac{x^*}{L}, u=\frac{u^*}{V}, p=\frac{p^*}{\rho V^2}, t=\frac{t^*V}{L}.$$

We consider the1D NSE in non-dimensional form [1], [6], [9], [17] as

$$u_t + uu_x = -p_x + \frac{1}{\mathrm{Re}}u_{xx} \qquad (1)$$

Where $x\in(a,b), t\in(t_o,t_f)$, $\mathrm{Re}=\frac{VL}{\upsilon}$.

Here $u(x,t)$ denotes velocity, subscripts denote partial differentiation, p_x is the pressure gradient. We consider pressure gradient as a time dependent exponential decreasing function of the form $-p_x = f(t) = e^{-at-b}$.

By considering - $p_x = f(t) = e^{-at-b}$ equation (1) reads as

$$u_t + uu_x = f(t) + cu_{xx} \tag{2}$$

Where $f(t) = e^{-at-b}$ and $c = \frac{1}{\mathrm{Re}} = \frac{\upsilon}{VL}$.

In order to determine numerical solution of equation (2) first we reduce it to Burgers equation by using OST. Then solving Burgers equation numerically we obtain the numerical solution of 1D NSE (2) by inverse OST.

The OST [4] is defined as

$$x' = x - \phi(t),\ t' = t,\ u'(x', t') = u(x,t) - W(t) \tag{3}$$

where $W(t) = \int_0^t f(\tau)d\tau = \frac{1}{a}\left(e^{-b} - e^{-at-b}\right)$.

and $\phi(t) = \int_0^t W(\tau)d\tau = \frac{1}{a}\left[te^{-b} + \frac{e^{-at-b}}{a} - \frac{e^{-b}}{a}\right]$.

Thus we get $x' = x - \frac{1}{a}te^{-b} - \frac{1}{a^2}e^{-at-b} + \frac{1}{a^2}e^{-b}$.

$$t' = t.$$

$$u'(x', t') = u(x,t) - \frac{1}{a}e^{-b} + \frac{1}{a}e^{-at-b}.$$

$$\frac{\partial u}{\partial t} = \frac{\partial u'}{\partial x'}\left(-\frac{1}{a}e^{-b} + \frac{1}{a}e^{-at-b}\right) + \frac{\partial u'}{\partial t'} + e^{-at-b}.$$

$$\frac{\partial u}{\partial x} = \frac{\partial u'}{\partial x'}.$$

Also $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u'}{\partial x'^2}$.

Substituting these transformed derivatives in equation (2) , we get

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} = c\frac{\partial^2 u'}{\partial x'^2} \tag{4}$$

Remarks 1: If we take c = 1/Re then we get from equation (4)

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} = \frac{1}{\mathrm{Re}}\frac{\partial^2 u'}{\partial x'^2}.$$

Which is 1D NSE in non-dimensional form after applying OST.This equation is analogous to non-dimensional form of 1D Burgers' equation

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} = \frac{1}{\mathrm{Re}}\frac{\partial^2 u^*}{\partial x^{*2}}.$$

3 EXPLICIT FINITE DIFFERENCE SCHEME FOR REDUCED 1D NSE

$\frac{\partial u'}{\partial t'} \approx \frac{u'^{n+1}_i - u'^{n}_i}{k}$. [The derivative of velocity versus time can be approximated with a first order forward finite difference approximation]

$\frac{\partial u'}{\partial x'} \approx \frac{u'^{n}_{i+1} - u'^{n}_{i-1}}{h}$. [The derivative of velocity versus space can be approximated with a first order centered finite difference approximation]

$\frac{\partial^2 u'}{\partial x'^2} \approx \frac{u'^{n}_{i+1} - 2u'^{n}_i + u'^{n}_{i-1}}{h^2}$. [The second order derivative of velocity versus space can be approximated with a second order centered finite difference approximation]

Here n represents the velocity at the current time step whereas (n+1) represents the new (future) velocity. The subscript i refer to the location. Both n and i are integers.

Substituting these approximations in equation (4), rearranging of the discretized equation so that all known quantities are on the right hand side and the unknown quantities on the left hand side we get the discretized version of the Burgers' equation in non-dimensional form.

$$u_i'^{n+1} = u_i'^{n} - \frac{ku_i'^{n}}{2h}\left(u'^{\,n}_{i+1} - u'^{\,n}_{i-1}\right) + \frac{ck}{h^2}\left(u'^{\,n}_{i+1} - 2u_i'^{n} + u'^{\,n}_{i-1}\right) \tag{5}$$

This is the explicit finite difference scheme for the the IBVP.

By using inverse OST, we obtain the explicit finite difference scheme for 1D NSE as by following page no. 328 in [12]

(4)

$$u_i^{n+1} - \frac{1}{a}e^{-b} + \frac{1}{a}e^{-ank-b} = u_i'^{n} - \frac{ku_i'^{n}}{2h}\left(u'^{\,n}_{i+1} - u'^{\,n}_{i-1}\right) + \frac{ck}{h^2}\left(u'^{\,n}_{i+1} - 2u_i'^{n} + u'^{\,n}_{i-1}\right)$$

$$\Rightarrow u_i^{n+1} = \frac{1}{a}e^{-b} - \frac{1}{a}e^{-ank-b} + u_i'^{n} - \frac{ku_i'^{n}}{2h}\left(u'^{\,n}_{i+1} - u'^{\,n}_{i-1}\right) + \frac{ck}{h^2}\left(u'^{\,n}_{i+1} - 2u_i'^{n} + u'^{\,n}_{i-1}\right) \tag{6}$$

4 STABILITY CONDITION FOR THE REDUCED 1D NSE

Here our aim is to investigate the stability condition of the explicit scheme in order to avoid oscillation and keep the scheme stable [5], [14], [15].

(5) can be written as

$$u_i'^{n+1} = \left(1-\frac{2ck}{h^2}\right)u_i'^n + \left(\frac{ck}{h^2}-\frac{ku_i'^n}{2h}\right)u_{i+1}'^n + \left(\frac{ck}{h^2}+\frac{ku_i'^n}{2h}\right)u_{i-1}'^n \qquad (7)$$

The new solution is a convex combination of the previous three solutions if

$$1-\frac{2ck}{h^2} \geq 0 \Rightarrow c \leq \frac{h^2}{2k} \qquad (8)$$

and $\frac{ck}{h^2}-\frac{ku_i'^n}{2h} > 0 \Rightarrow c > \frac{h}{2}u_i'^n$ (9)

and $\frac{ck}{h^2}+\frac{k}{2h}u_i'^n > 0 \Rightarrow c > -\frac{h}{2}u_i'^n$ (10)

Since $\max_i\{u_i'^0\} \geq u_i'^n$, so (9) can be true if

$$c > \frac{h}{2}\max_i\{u_i'^0\} \qquad (11)$$

Again since $\min_i\{u_i'^0\} \leq u_i'^n$, so (10) can be true if

$$c > \frac{h}{2}\min_i\{u_i'^0\} = \frac{h}{2}\max_i\{-u_i'^0\}$$

$$\Rightarrow c > \frac{h}{2}\max_i\{-u_i'^0\} \qquad (12)$$

From (11) and (12), we have

$$c > \frac{h}{2}\max_i\{|u_i'^0|\} \qquad (13)$$

So if $\frac{h}{2}\max_i\{|u_i'^0|\} < c \leq \frac{h^2}{2k}$ (14)
then the new solution is a convex combination of the three previous solutions. That is the solution at a new time-step (n+1) at a spatial node i, is an average of the solution at the previous time-step at the spatial nodes i,i+1,i-1. This means that the extreme value of the new solution is the average of the extreme values of the previous solution at the three consecutive nodes.

Now applying inverse OST we get from (14)

$$\frac{h}{2}\max_i\{|u_i^0|\} < c \leq \frac{h^2}{2k} \qquad (15)$$

This is the stability condition for reduced 1D NSE which has to be satisfied in order to get well-behaved numerical approximations.

We collect the above result as follows:

Theorem: The stability condition of the reduced1D NSE is the same as that of 1D viscous Burgers equation.

5 NUMERICAL IMPLEMENTATION OF THE EXPLICIT FINITE DIFFERENCE SCHEME

To implement our scheme, we consider the spatial domain [0,2π] and the maximum time step T=9;

We consider the initial condition

$u(x,0)=u_0(x)=\sin(x)$

and the homogeneous Dirichlet boundary condition

$$u(0,t) = u(2\pi,t)\frac{1}{a}e^{-b} - \frac{1}{a}e^{-at-b}$$

For c = 0.1, we get the stability condition

$$\frac{h}{2}\max_i\{|u_i^0|\} < 0.1 \leq \frac{h^2}{2k}$$

And the stability condition becomes

$$\frac{h}{2} < 0.1 \leq \frac{h^2}{2k}.$$

If we take m =50 and n = 450 then we get

$$\frac{h}{2} = 0.062831853071796, \frac{h^2}{2k} = 0.296088132032682$$

and this values of h and k satisfy our stability condition.

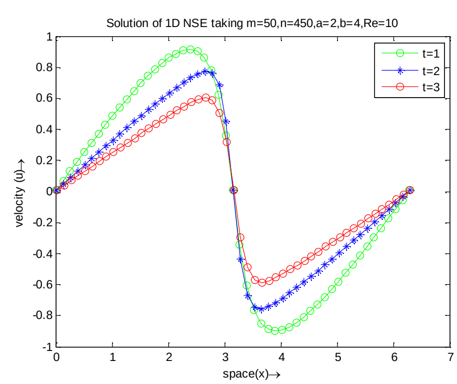

Fig. 1. Solution of 1D NSE using explicit finite difference scheme at different time steps.

Fig. 1 indicates that our solution profile in three different time steps agrees with the convection-diffusion type qualitative behavior. That is, the solution profile is moving forward (convection) and smears out (diffusion) as well.

M.A.K. Azad, L.S. Andallah [17] derived an analytical solution for 1D NSE as

$$u(x,t)=\frac{1}{a}e^{-b}-\frac{1}{a}e^{-at-b}+\frac{\int_{-\infty}^{\infty}\left(x-\frac{1}{a}te^{-b}-\frac{1}{a^2}e^{-at-b}+\frac{1}{a^2}e^{-b}-y\right)\exp\left[-\frac{\left(x-\frac{1}{a}te^{-b}-\frac{1}{a^2}e^{-at-b}+\frac{1}{a^2}e^{-b}-y\right)^2}{4ct}-\frac{1}{2\upsilon}\int_0^y u_0'(z)dz\right]dy}{t\int_{-\infty}^{\infty}\exp\left[-\frac{\left(x-\frac{1}{a}te^{-b}-\frac{1}{a^2}e^{-at-b}+\frac{1}{a^2}e^{-b}-y\right)^2}{4ct}-\frac{1}{2c}\int_0^y u_0'(z)dz\right]dy} \quad (16)$$

Here it is very important that for very small t, both numerator and denominator of " (16)" get very closed to zero and thus very difficult to handle numerically. Again, for very small c, both numerator and denominator of "(16)" get very closed to zero or infinity which becomes very difficult to handle.

Fig. 2, fig.3&fig.4 show the comparison of analytical and numerical solution of 1D Navier-Stokes equation in the domain T=[0,3],x=[0,2π].

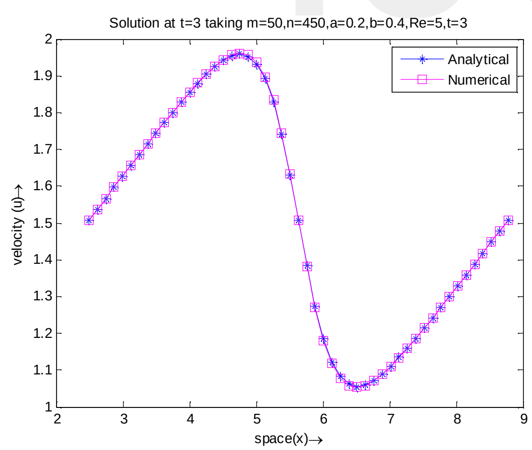

Fig. 2. Comparison of numerical and analytical solution of 1D NSE.

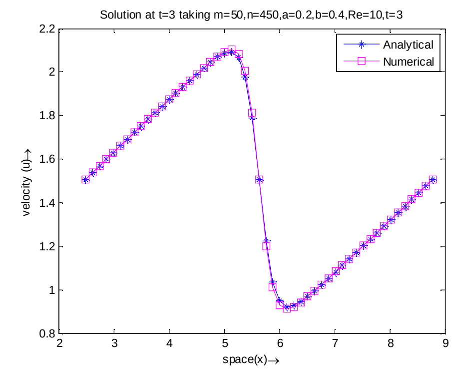

Fig. 3. Comparison of numerical and analytical solution of 1D NSE.

We have compared the solution graphically from the following figure taking L=2π, T=3.

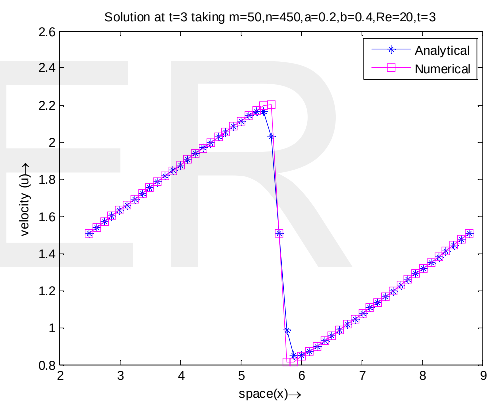

Fig. 4. Comparison of numerical and analytical solution of 1D NSE.

Fig.2, fig.3, fig.4 show a very good agreement between analytical and numerical solution at Re=5, Re=10&Re=20 respectively.

6 RELATIVE ERROR ESTIMATION FOR REDUCED 1D NSE

We compute the relative error in L_1-norm defined by

$$\|e\|_1=\frac{\|u_e-u_n\|}{\|u_e\|_1}.$$

Where u_e is the exact solution and u_n is the numerical solution computed for the finite difference scheme.

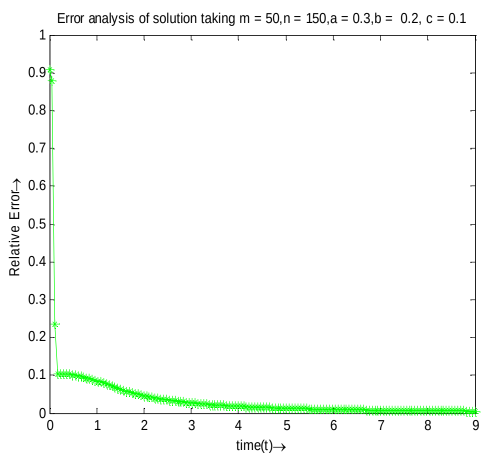

Fig. 5

Here it is important that for 0<t<0.18 analytical solution is very difficult to handle. But numerical solution using finite difference scheme is easy. So, for first three time steps relative errors are very high. But 0.18<t<9 relative errors level are quite satisfactory.

After computation of relative errors, we show the convergence of the scheme by plotting relative errors for different pairs of (h,k).We perform our numerical scheme for c = 0.2 up to time t = 12 in spatial domain $[0,2\pi]$ and taking m =40, n = 300 we get the stability condition and we get relative errors which are shown in fig. 6.

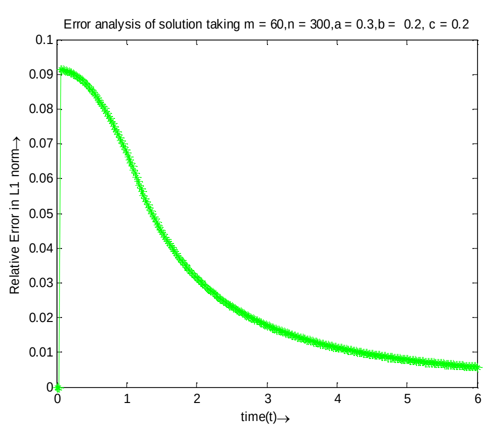

Fig. 6.

Fig. 6. Shows that the relative error in this case remains below 0.090 and above 0.00058 .The relative errors are quite acceptable. Here it is important that for 0<t<0.12 analytical solution is very difficult to handle. But numerical solution using finite difference scheme is easy. But 0.16<t<12 relative errors level are quite satisfactory.

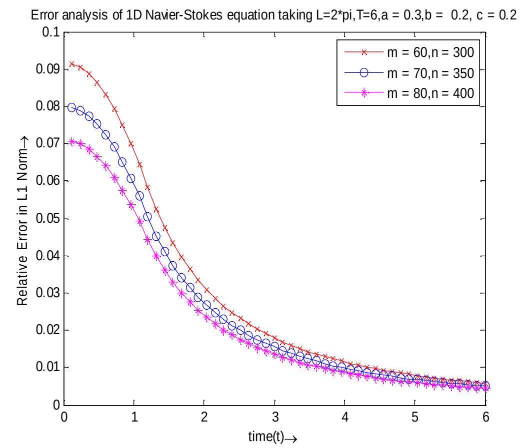

Fig. 7. Relative errors and convergence for the FDS for the reduced 1D NSE.

From fig.7. it is observed that relative error is from 0.0707 to 0.0913 at t=0.12 which is high. It is happened for small value of t. For 0.12<t<1 relative errors range is from 0.0493 to 0.0645 which seems somewhat high. But for 1<t<2 this range is from 0.0234 to 0.0334. When 2<t<3 the observed range is from 0.0136 to 0.0179 which is acceptable. It is also studied that in range 3<t<6 the relative errors remain from 0.0044 to 0.0203 which is quite acceptable.

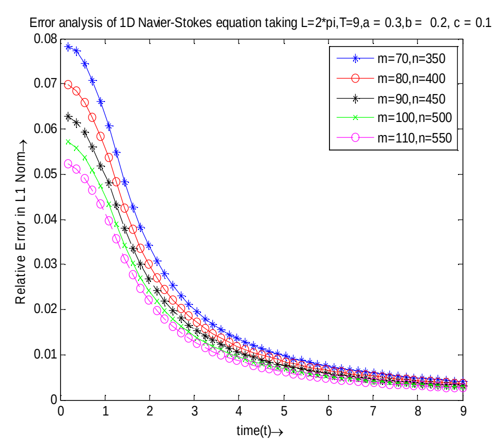

Fig. 8. Relative errors and convergence for the FDS for the reduced 1D NSE.

From fig.8. it is observed that relative error is from 0.0532 to 0.0782 at t=0.18 which is somewhat high. It is happened for small value of t. For 0.18<t<1 relative errors range is from 0.0396 to 0.0772. But for 1<t<2 this range is from 0.0221 to 0.0548. When 2<t<3 the observed range is from 0.0125 to 0.0342 which is acceptable. It is also studied that in range 3<t<9 the relative errors remain from 0.0048 to 0.0180 which is quite acceptable.

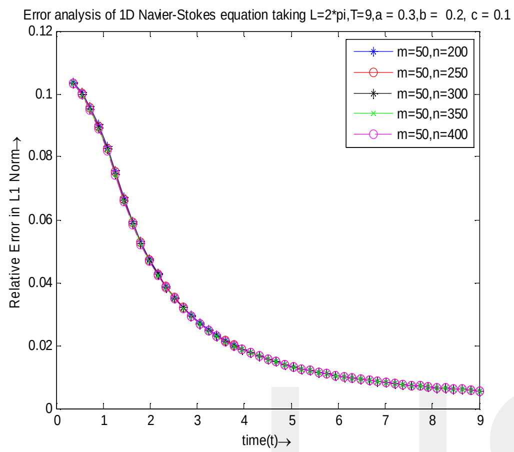

Fig. 9. Relative errors and convergence for the FDS.

Fig.9. shows that keeping spatial steps fixed if time steps increase then relative errors decrease but very close to each other. So, in this case it seems relative errors coincide to each other for each pair (m,n). It is also observed that for t<1, relative errors are somewhat high. But for 2<t<9 estimated relative errors are quite acceptable. So, our findings in this case are n=200 are good enough to restrict the error due to the time discretization.

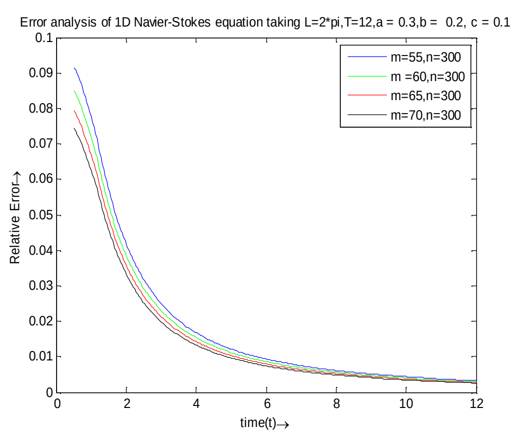

Fig. 10. Relative errors and convergence for the FDS.

From fig.10. it is clear that keeping no. of time steps fixed and increasing no. of spatial steps relative errors are decreasing.

After computation of relative errors, we show the convergence of each scheme by plotting relative errors for different pairs of (h,k). We perform our numerical scheme for c = 0.1, 0.2, 0.2, c=0.1 up to time t = 9, 12,6,12 respectively in spatial domain $[0,2\pi]$ and taking different spatial steps and time steps we get the stability condition. From fig. 7, 8, 9 and 10 we observe that the calculated relative errors are quite acceptable. We see that relative errors are decreasing with respect to the smaller discretization parameters h and k which show the convergence of the explicit finite difference scheme.

7 PRESSURE GRADIENT EFFECT ANALYSIS

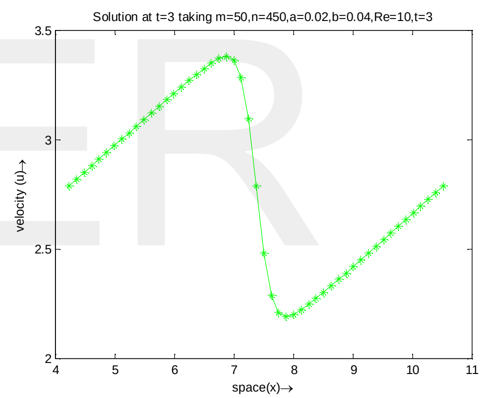

Fig.11. Numerical solution of 1D NSE.

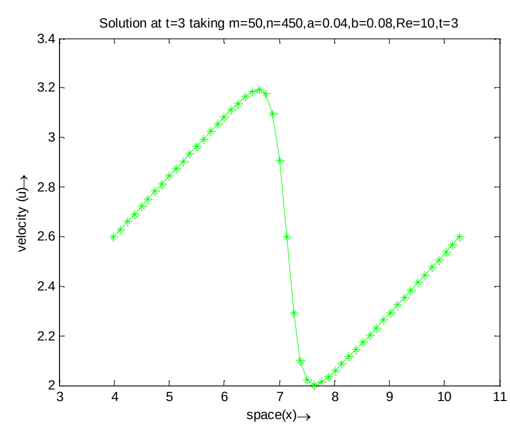

Fig. 12. Numerical solution of 1D NSE.

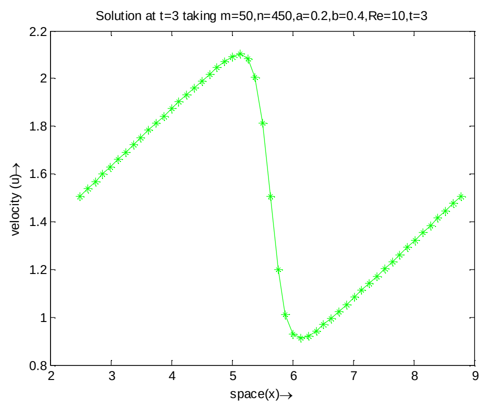

Fig.13. Numerical solution of 1D NSE.

Comparing above figures we conclude that pressure gradient term has an important effect on the solution of 1D NSE. From the above fig. 11, fig. 12, fig. 13 we observe that an increase in the pressure gradient resulted an increase in the velocity of the fluid. Also a decrease in the pressure gradient resulted a decrease in the velocity of the fluid which describes a phenomenon in which the pressure of a fluid changes with a change in the velocity of the fluid. This qualitative behavior agrees with the qualitative behavior obtained from the numerical evaluation of analytical solution which is shown by M.A.K. Azad and L.S. Andallah in [17].

8 REYNOLDS NUMBER EFFECT ANALYSIS

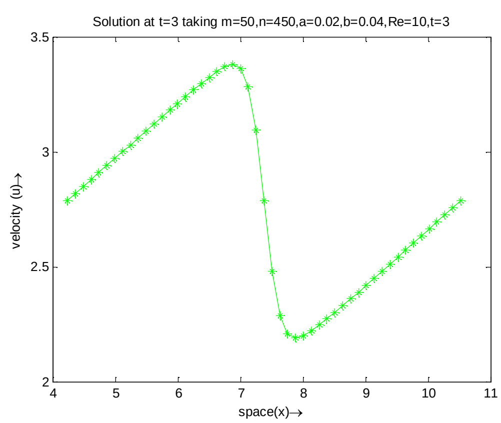

Fig. 14. A numerical solution of 1D NSE.

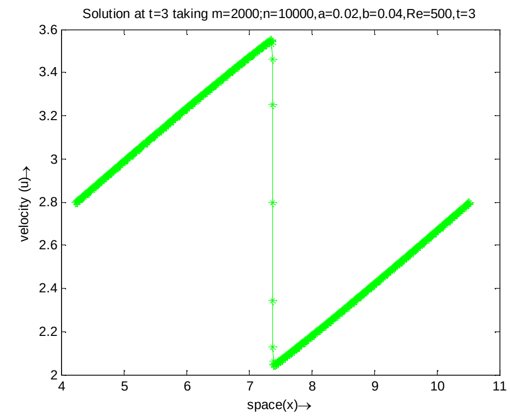

Fig. 15. A numerical solution of 1D NSE.

From fig.14, fig.15 we observe that our initial date is smooth and c is very small i.e. Reynolds number is very large, then before the wave begins to break and shock forms. This qualitative behavior agrees with the qualitative behavior obtained from the numerical evaluation of analytical solution which is shown by M.A.K. Azad and L.S. Andallah in [17].

9 FINITE DIFFERENCE SCHEME FOR THE ORIGINAL 1D NSE

$$u_t \approx \frac{u_i^{j+1} - u_i^j}{k}$$

$$u_x \approx \frac{u_{i+1}^j - u_{i-1}^j}{2h}$$

$$u_{xx} \approx \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}$$

$$p_x \approx \frac{1}{a} e^{-ajk-b}$$

Substituting these approximations, we obtain

$$u_i^{j+1} = ke^{-ajk-b} + u_i^j + u_i^j\left(u_{i+1}^j - u_{i-1}^j\right)_t + \frac{ck}{h^2}\left(u_{i+1}^j - 2u_i^j + u_{i-1}^j\right)$$

This is the explicit finite difference scheme for the original NSE. But the new solution is not a convex combination of the other three previous solutions. So numerical experiment is required for the stability condition of this scheme.

10 COMPARISON OF EFFECTIVENESS OF THE SCHEMES

We perform our numerical scheme for Re =10 up to time t = 9 in spatial domain [0,2π] and taking different spatial and time steps which satisfy stability condition and we get computational time by using MAT LAB program in HP core i5 model laptop which are tabulated in the following table : Table 1

No. of spatial nodes (m)	No. of time steps(n)	Numerical scheme for original 1D NSE	Numerical scheme for reduced 1D NSE	Percentage of saving time (%)
70	350	0.011229 s	0.010485 s	6.625701
80	400	0.012501 s	0.010969 s	12.25502
90	450	0.012758 s	0.011463 s	10.150494
100	500	0.013844 s	0.012248 s	10.529788
110	550	0.015157 s	0.013185 s	13.010490

Table 1 show that both schemes are very fast. The speed for implementing the numerical scheme for reduced NSE is faster than the speed for implanting the numerical scheme for the original NSE. So, numerical scheme for reduced NSE is more efficient than numerical scheme for original NSE

11 CONCLUSIONS

In this paper, we have presented the numerical solution of 1D NSE by using finite difference scheme for the reduced 1D NSE. Computational results obtained from numerical solution of 1D NSE by implementing computer program we have found some qualitative behaviors which agree with the qualitative behaviors obtained from analytical solution.

The stability condition of the reduced model Burgers equation is the same as that of the original Burgers equation. However, the finite difference scheme for the original 1D NSE requires much smaller time step size selection for the stability condition. Our numerical experiment shows that time step size is 68.1691 % smaller than the reduced model. We have computed relative errors which show a good rate of convergence of the numerical scheme. Our experiment also reveals that numerical scheme for the reduced NSE is faster than direct numerical scheme for original 1D NSE. Thus we conclude that the finite difference scheme for reduced model is much more efficient than the finite difference scheme for the original NSE.

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